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A COUNTER-EXAMPLE TO A CONJECTURE OF FRIEDLAND

BY

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ABSTRACT

In 1982, S. Friedland proved that a bounded linear operator A on a Hilbert space is normal if and only if

$$(\alpha I + A + A^*)^2 \geqq AA^* - A^*A \geqq -(\alpha I + A + A^*)^2 \text{ for all real } \alpha.$$

And he conjectured the inequality $(\alpha I + A + A^*)^2 \geqq AA^* - A^*A$ for all real α will imply that $A^*A - AA^* \geqq 0$, i.e., A is hyponormal. But his conjecture is incorrect. In this note I'll give a counter-example for his conjecture.

THEOREM: *There is a non-hyponormal operator A which satisfies the inequality $(\alpha I + A + A^*)^2 \geqq AA^* - A^*A$ for all real α.*

Proof: For an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of a Hilbert space H, let $Ae_0 = ae_1$ and $Ae_n = e_{n+1} (n = 1, 2, \dots)$ where

$$1 < a \leqq \sqrt{\frac{5 - 2\sqrt{2}}{2}}.$$

Then A is not hyponormal as for any $x = \Sigma_{n=0}^{\infty} \lambda_n e_n \in H$ the following equality holds:

$$\begin{aligned}\langle (AA^* - A^*A)x, x\rangle &= \|A^*x\|^2 - \|Ax\|^2 \\ &= \|a\lambda_1 e_0 + \lambda_2 e_1 + \cdots\|^2 - \|a\lambda_0 e_1 + \lambda_1 e_2 + \cdots\|^2 \\ &= \{a^2|\lambda_1|^2 + |\lambda_2|^2 + \cdots\} - \{a^2|\lambda_0|^2 + |\lambda_1|^2 + \cdots\} \\ &= -a^2|\lambda_0|^2 + (a^2 - 1)|\lambda_1|^2 \qquad (1)\end{aligned}$$

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and $a > 1$.

Since

$$\begin{aligned}
&\|(\alpha I + A + A^*)x\|^2 \\
&= \|\alpha \sum_{n=0}^{\infty} \lambda_n e_n + a\lambda_1 e_0 + (a\lambda_0 + \lambda_2)e_1 + \cdots\|^2 \\
&= \|(\alpha\lambda_0 + a\lambda_1)e_0 + (a\lambda_0 + \alpha\lambda_1 + \lambda_2)e_1 + (\lambda_1 + \alpha\lambda_2 + \lambda_3)e_2 + \cdots\|^2 \\
&= |\alpha\lambda_0 + a\lambda_1|^2 + |a\lambda_0 + \alpha\lambda_1 + \lambda_2|^2 + |\lambda_1 + \alpha\lambda_2 + \lambda_3|^2 + \cdots,
\end{aligned}$$

we have

$$\begin{aligned}
&\langle(\alpha I + A + A^*)^2 x, x\rangle - \langle(AA^* - A^*A)x, x\rangle \\
&\qquad \geqq |\alpha\lambda_0 + a\lambda_1|^2 + a^2|\lambda_0|^2 - (a^2 - 1)|\lambda_1|^2 \quad \text{by (1)} \\
&\qquad = (\alpha^2 + a^2)|\lambda_0|^2 + \alpha a(\lambda_0\bar{\lambda}_1 + \bar{\lambda}_0\lambda_1) + |\lambda_1|^2. \qquad (2)
\end{aligned}$$

If $|\alpha| \leqq a/\sqrt{a^2 - 1}$, then $\alpha^2 a^2 \leqq \alpha^2 + a^2$ and hence we have

$$\begin{aligned}
&\langle(\alpha I + A + A^*)^2 x, x\rangle - \langle(AA^* - A^*A)x, x\rangle \\
&\qquad \geqq \alpha^2 a^2|\lambda_0|^2 + \alpha a(\lambda_0\bar{\lambda}_1 + \bar{\lambda}_0\lambda_1) + |\lambda_1|^2 \quad \text{by (2)} \\
&\qquad = |\alpha a\lambda_0 + \lambda_1|^2 \geqq 0.
\end{aligned}$$

Therefore

$$(3) \quad (\alpha I + A + A^*)^2 \geqq AA^* - A^*A \text{ for all real } \alpha \text{ such as } |\alpha| \leqq \frac{a}{\sqrt{a^2 - 1}}.$$

Next we have $\|(\alpha I + A + A^*)x\| \geqq |\alpha|\,\|x\| - \|A + A^*\|\,\|x\| \geqq (|\alpha| - 2a)\|x\|$ because $\|A\| = a$ and the assumption

$$1 < a \leqq \sqrt{\frac{5 - 2\sqrt{2}}{2}}$$

implies that

$$0 < \sqrt{a^2 - 1} \leqq 1 - \frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{a}{\sqrt{a^2 - 1}} \geqq \frac{a}{1 - \frac{\sqrt{2}}{2}} = (2 + \sqrt{2})a.$$

And then, for each α such as $|\alpha| > \frac{a}{\sqrt{a^2-1}}$, we have

$$\|(\alpha I + A + A^*)x\| \geqq (\frac{a}{\sqrt{a^2 - 1}} - 2a)\|x\| = \sqrt{2}a\|x\|$$

and

$$\begin{aligned}\langle(\alpha I + A + A^*)^2 x, x\rangle &= \|(\alpha I + A + A^*)x\|^2 \geqq 2a^2\|x\|^2 \\ &\geqq \|AA^* - A^*A\|\ \|x\|^2 \quad \text{because } \|A\| = a \\ &\geqq \langle(AA^* - A^*A)x, x\rangle.\end{aligned}$$

Hence

$$(4) \qquad (\alpha I + A + A^*)^2 \geqq AA^* - A^*A \quad \text{for all real } \alpha \text{ such as } |\alpha| > \frac{a}{\sqrt{a^2-1}}.$$

By (3) and (4), we have $(\alpha I + A + A^*)^2 \geqq AA^* - A^*A$ for all real α.

References

1. S. Friedland, *A characterization of normal operators*, Isr. J. Math. **42** (1982), 235–240.